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A Class of Probabilistic Unfolding Models for Polytomous Responses

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By revisiting the approaches used to present the Rasch model for polytomous response, this paper uses the principle of the *rating formulation* (Andrich, 1978) to construct a class of unfolding models for polytomous responses in terms of a set of latent dichotomous unfolding variables. By anchoring the dichotomous unfolding variables involved at the same location, this paper presents a formulation of a very general class of unfolding models for ordered polytomous responses, of which the unfolding models for ordered polytomous responses proposed hitherto are special cases. Within this class, the analytic and measurement properties of the probabilistic functions are well interpreted in terms of the latitudes of acceptance parameters of the dichotomous unfolding models. Based on the general form of this class of unfolding models, some new models are readily specified.

1. INTRODUCTION

The collections of direct responses of persons to stimuli are known as *single stimulus* data (Coombs, 1964). Single stimulus data can be coded in two forms: dichotomous and polytomous. In the context of attitude measurement, the former has only two response categories: 0, *Disagree*, and 1, *Agree*. The latter allows for more than two ordered response categories; for example, 0, *Strongly Disagree*, 1, *Disagree*, 2, *Agree*, and 3, *Strongly Agree*. Coombs (1950, 1964) used the term *unfolding technique* to describe a procedure to construct a joint scale (*J*-scale) from a set of individual scales (*I*-scale). Coombs and Avrunin (1977) pointed out that single-peaked functions are the foundation underlying unfolding theory. In the literature, the response process in which a single-peaked function is dominant is generally termed an *unfolding process*. The models conforming to the unfolding process are classified as unfolding models.

The last decade or two has seen a major interest in the development of explicit probabilistic unfolding models for dichotomous responses (e.g., DeSarbo & Hoffman,

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1986; Andrich, 1988, 1989, 1996; Hoijtink, 1990, 1991; Andrich & Luo, 1993; Verhelst & Verstralen, 1993). Luo (1998a) proposed a general form of the probabilistic unfolding models for dichotomous responses, of which all the previous proposed probabilistic dichotomous unfolding models are special cases.

Though data collected in the polytomous format are “ubiquitous” (Dawes, 1972) in attitude measurement, and though it is generally suggested that precision is greater with polytomous responses than with dichotomous responses, the development of unfolding models for polytomous responses has lagged behind the demand that arises from real applications. In the last few years, some models for the polytomous unfolding responses have been proposed (Andrich, 1996; Roberts & Laughlin, 1996; Rost & Luo, 1997). A basic conceptualization on polytomous unfolding models is that the mathematical expectation of the response variable is a single-peaked function of the person location parameter involved. However, the underlying structure of the unfolding models for polytomous responses is yet to be identified.

The purpose of this paper is to explore the structure of unfolding models for polytomous responses by introducing a general form for such models. To elucidate the motivation and development of this paper, the background of this paper is reviewed briefly.

1.1. The Development of the General Form of the Probabilistic Unfolding Models for Dichotomous Responses

In the context of unidimensional unfolding models, persons and statements (items) involved in an attitude measurement are envisaged being located on a real line termed the latent latitude continuum. Among the specific unfolding models in the literature, the hyperbolic cosine model (HCM) (Andrich & Luo, 1993) is distinguished by its construction and its structure. In constructing the HCM, the *Disagree* response was resolved into two latent components, *Disagree below*, which reflects that the person may be located below the statement, and *Disagree above*, which reflects that the person may be located above the statement. Then the Rasch model for three ordered response categories was applied to these two components of the *Disagree* response together with the single *Agree* response. The final form of the HCM was obtained by summing the probabilities of the two latent components of the *Disagree* response to reflect the single manifest *Disagree* response. Suppose that N persons give responses to a questionnaire consisting of I statements (items). For any person n and any item i, the response variable is denoted as $X_{ni} : x_{ni} \in \{0, 1\}$. According to the HCM, the probability that person n gives an *Agree* response to item i is

$$\Pr\{X_{ni} = 1 \mid \beta_n, \delta_i, \theta_i\} = \frac{\exp(\theta_i)}{\exp(\theta_i) + 2\cosh(\beta_n - \delta_i)}, \tag{1}$$

where β_n is the location parameter for person n, δ_i is the location parameter for item i (β_n and δ_i can be any real number), and θ_i ($\geqslant 0$) is the unit parameter for

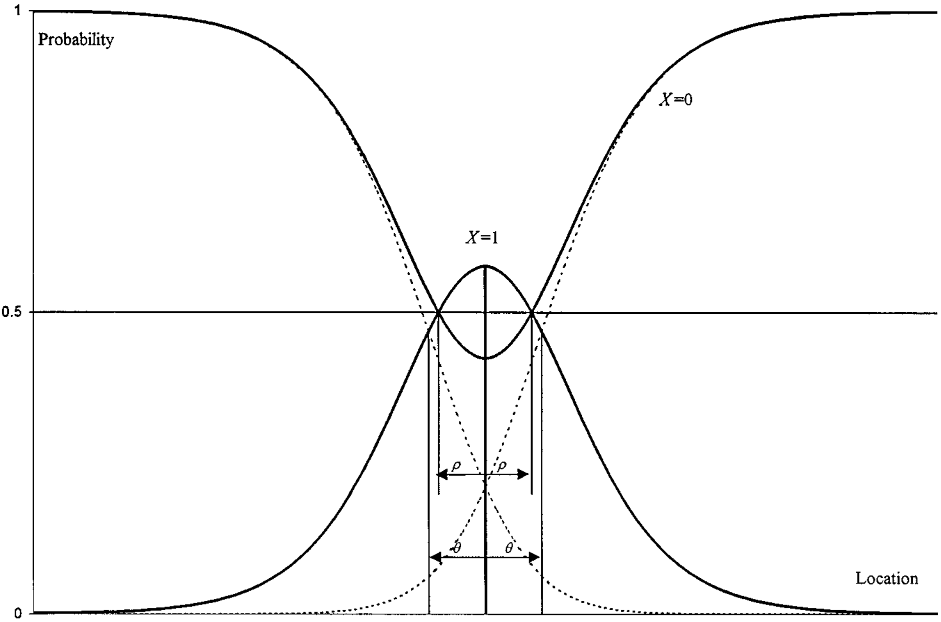

FIG. 1. The probabilistic functions of the HCM including the resolved components of the *Disagree* response.

item i ($n = 1, ..., N$, $i = 1, ..., I$). Figure 1 shows the probabilistic functions of the HCM.

The distinguishing structure of the HCM of Eq. (1) is that in addition to the item location δ_i, a second item parameter θ_i appears as a unit parameter. Luo (1998a) reparameterised the HCM into the following form, which involves only the hyperbolic cosine function:

$$\Pr\{X_{ni} = 1 \mid \beta_n, \delta_i, \rho_i\} = \frac{\cosh(\rho_i)}{\cosh(\rho_i) + \cosh(\beta_n - \delta_i)}, \tag{2}$$

where the parameter $\rho_i (\geqslant 0)$ reflects the *latitude of acceptance*, an important concept in attitude measurement (Sherif & Sherif, 1967). As shown in Fig. 1, $(\delta_i \pm \rho_i)$ are the two intersection points of the positive and negative response curves. The relationship between θ_i and ρ_i is

$$e^{\theta_i} = 2\cosh(\rho_i). \tag{3}$$

It is noted that Eq. (3) is valid only when $\theta_i \geqslant \ln 2$. (When $\theta_i < \ln 2$, the probability for a positive response is always less than that for a negative response regardless of the locations of the person and the item. Model (1) is considered invalid in this case.) Based on the development of the HCM, software for analysing dichotomous unfolding data has been made available for the real applications (Andrich & Luo, 1998).

Furthermore, by making an abstraction from the reparameterised form of the HCM (2), Luo (1998a) presented the general form of the probabilistic unfolding models for dichotomous responses as follows,

$$\Pr\{X_{ni}=1 \mid \beta_n, \delta_i, \rho_i\} = \frac{\Psi(\rho_i)}{\Psi(\rho_i)+\Psi(\beta_n-\delta_i)}, \tag{4}$$

where β_n, δ_i, and ρ_i are as defined in (2), and the function Ψ was termed the *operational function*. It has the following properties:

(P1) *Non-negative*: $\Psi(t) \geqslant 0$ for any real t;

(P2) *Monotonic in the positive domain*: $\Psi(t_1) > \Psi(t_2)$ for any $t_1 > t_2 > 0$; and

(P3) Ψ *is an even function* (*symmetric about the origin*): $\Psi(t) = \Psi(-t)$ for any real t.

The generality of (4) was consolidated by the fact that when the operational function was specified to be the hyperbolic cosine function, the square exponential function, and the square function respectively, (4) led to corresponding unfolding models previously proposed in the literature. The implied latitude of acceptance parameters in these models were also identified in Luo (1998a). The JML procedure for the parameter estimation of the general dichotomous unfolding model (4) is proposed by Luo, Andrich, and Styles (1998).

1.2. The Developments of Probabilistic Cumulative Models for Polytomous Responses

Among the probabilistic models for polytomous responses, the Rasch model for polytomous responses is distinguished by its constructional features along with its fundamental properties. It was constructed rigorously from the simple Rasch model (Rasch, 1961) or one-parameter logistic model. In the simple Rasch model, a dichotomous response X_{ni} is governed by only one item location parameter δ_i and one person location parameter β_n:

$$\Pr\{X_{ni}=1 \mid \beta_n, \delta_i\} = \frac{\exp(\beta_n-\delta_i)}{1+\exp(\beta_n-\delta_i)}. \tag{5}$$

The rating formulation. Andrich (1978) constructed the Rasch model for ordered categories by introducing a series of dichotomous Rasch response variables $(Z_1, Z_2, ..., Z_m)$ with locations $\{\delta_1, \delta_2, ..., \delta_m\}$. A rationale in constructing and making meaning of the model was that these locations were ordered:

$$\delta_1 < \delta_2 < \cdots < \delta_k < \cdots < \delta_m. \tag{6}$$

When $(Z_1, Z_2, ..., Z_m)$ are independent, the entire sample space Ω includes 2^m response patterns. Ω' denotes the collection of the Guttman response patterns in Ω. For a polytomous response variable $X: x \in \{0, 1, 2, ..., m\}$, the one-to-one correspondence between the observations of X and Ω' was defined as follows:

$$\begin{aligned} X=0 \quad & \Leftrightarrow (0, 0, 0, ..., 0, 0), \\ X=1 \quad & \Leftrightarrow (1, 0, 0, ..., 0, 0), \\ & \cdots\cdots\cdots\cdots \\ X=m-1 & \Leftrightarrow (1, 1, 1, ..., 1, 0), \\ X=m \quad & \Leftrightarrow (1, 1, 1, ..., 1, 1). \end{aligned} \tag{7}$$

Then the probability $\Pr\{X=k\}$ is defined as the conditional probability of the corresponding pattern $(\underbrace{1, 1, ..., 1}_{k}, \underbrace{0, ..., 0}_{m-k}\}$ on the constrained space Ω'. When all the dichotomous response variables $\{Z_k, k=1, ..., m\}$ follow the simple Rasch model with locations $\{\delta_1, \delta_2, ..., \delta_m\}$, the induced variable X of successive categories follows the Rasch model for polytomous responses:

$$\Pr\{X=x \mid \beta_n, (\delta_k)\} = \frac{\exp\{\sum_{k=0}^{x} (\beta_n - \delta_k)\}}{\sum_{l=0}^{m} \exp\{\sum_{k=0}^{l} (\beta_n - \delta_k)\}}, \tag{8}$$

where for notational convenience, $\sum_{k=0}^{0} (\beta_n - \delta_k) \equiv 0$. Equation (8) was further simplified by parameterising the mean of $\{\delta_1, \delta_2, ..., \delta_m\}$ as the location of the polytomous variable X,

$$\delta = \frac{1}{m} \sum_{k=1}^{m} \delta_k, \tag{9}$$

and the deviations $\{\tau_k = \delta_k - \delta, k=1, ..., m\}$ as the (centralised) thresholds of the polytomous response model. Then the model has the form

$$\Pr\{X=x \mid \beta_n, \delta, (\tau_k)\} = \frac{\exp\{x(\beta_n - \delta) - \sum_{k=0}^{x} \tau_k\}}{\sum_{l=0}^{m} \exp\{l(\beta_n - \delta) - \sum_{k=0}^{l} \tau_k\}}, \tag{10}$$

where for notational convenience, $\tau_0 \equiv 0$. Figure 2 shows the probabilistic functions of (10) and those of the corresponding dichotomous Rasch variables $(Z_1, Z_2, ..., Z_m)$ in the case of $m=3$. It can be seen graphically that for $k=1, ..., m$, the crossing point of the probabilistic functions for adjacent categories $k-1$ and k are the locations of the corresponding dichotomous variable Z_k, which is $\delta + \tau_k$.

In the derivation above, X is a response variable on a particular item. To identify the response X with respect to item i with the maximum score of $m_i (i=1, ..., I)$, Eq. (8) can be written as

$$\Pr\{X_{ni} = x\} = \frac{\exp[\sum_{k=0}^{x} (\beta_n - \delta_{ik})]}{\sum_{l=0}^{m_i} \exp[\sum_{k=0}^{l} (\beta_n - \delta_{ik})]}, \qquad x = 0, 1, ..., m_i, \tag{11}$$

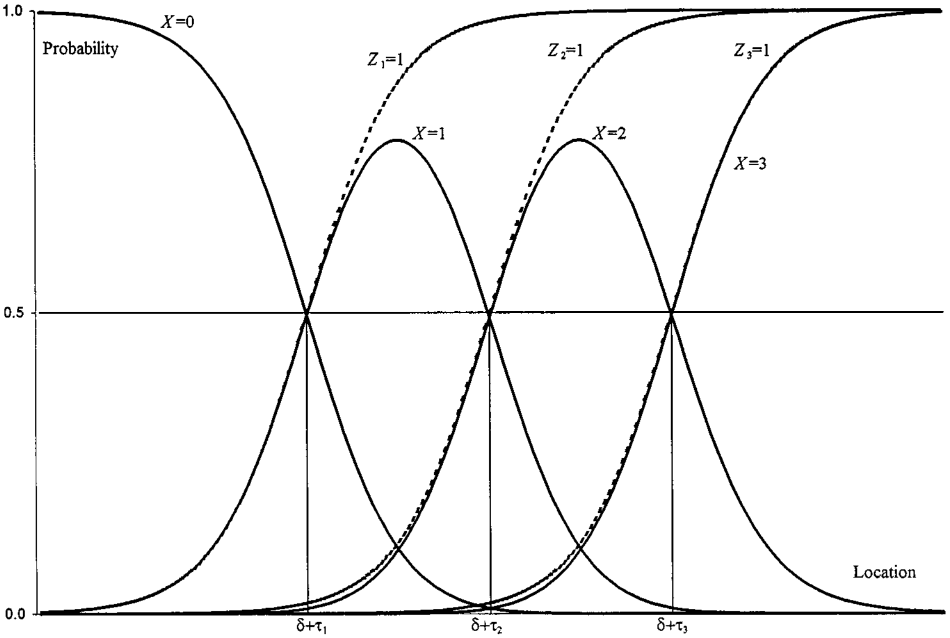

FIG. 2. The probabilistic functions of the Rasch model for polytomous responses.

where for notational convenience $\sum_{k=0}^{0}(\beta_n-\delta_{ik})\equiv 0$. With the constraint that the distance between successive thresholds is equal (and positive), Model (11) was further reparameterised into the form in which the probabilistic function is expressed in terms of item parameters including location δ_i, unit θ_i (Andrich, 1982):

$$\Pr\{X_{ni}=x \mid \beta_n, \delta_i, \theta_i\}=\frac{\exp\{x(\beta_n-\delta_i)+x(m_i-x)\,\theta_i\}}{\sum_{k=0}^{m_i}\exp\{k(\beta_n-\delta_i)+k(m_i-k)\,\theta_i\}}. \tag{12}$$

(In the rest of this paper, the parameters on the left-hand side of the expression as above are often omitted and the conditional probability is simply written as $\Pr\{X=x\}$ when this would not cause confusion, or in some situations, only some of the parameters are listed for the purpose of emphasis.)

It is easy to see now that according to Eq. (11), for person n and item i with maximum score m_i, the conditional probability for any pair of adjacent response categories takes the form of the dichotomous Rasch model. That is,

$$\Phi_{kni}=\frac{\Pr\{X_{ni}=K\}}{\Pr\{X_{ni}=K-1\}+\Pr\{X_{ni}=K\}}=\frac{\exp\{\beta_n-\delta_{ik}\}}{1+\exp\{\beta_n-\delta_{ik}\}}, \qquad k=1, ..., m_i. \tag{13}$$

In fact, Masters (1982) began with Eq. (13) and obtained the model of Eq. (11) with the required constraint

$$\sum_{k=0}^{m_i}\Pr\{X_{ni}=k\}=1. \tag{14}$$

Though Masters (1982) derived the same model as Andrich (1978), he argued that to make sense of the model of Eq. (11), the order of the locations $\{\delta_{i1}, \delta_{i2}, ..., \delta_{im_i}\}$ was not required, which contrasts with the structural requirement (6) of Andrich's rating formulation. (This issue will be broached again later in this paper in the context of formalising unfolding models for polytomous responses.)

1.3. Some Established Probabilistic Unfolding Models for Polytomous Responses

In the case that the response format is 0, *Strongly Disagree*; 1, *Disagree*; 2, *Agree*; and 3, *Strongly Agree*; Andrich (1996) and Rost and Luo (1997) resolved all the response categories except 3, *Strongly Agree* into two latent components respectively, one reflecting that the person may be located below the location of the statement and the other reflecting that the person may be located above the location of the statement. In the general case of $m+1$ ordered categories ($x_{ni}=0, 1, ..., m$), there are $(2m_i+1)$ resolved categories. The Rasch model for polytomous responses (Eq. (12)) was applied to this resolved format, and after the pairs of the latent components were summed to reflect the corresponding manifest category, the generalised hyperbolic cosine model (GHCM) took the form

$$\Pr\{X_{ni}=k\} = \frac{\exp[k(2m-k)\,\theta_i]\,2\cosh[(m-k)(\beta_n-\delta_i)]}{\gamma_{ni}}, \qquad k=0, ..., m-1,$$

$$\Pr\{X_{ni}=m\} = \frac{\exp[m^2\theta_i]}{\gamma_{ni}}, \tag{15}$$

where β_n, δ_i, and θ_i are as defined in (12), and γ_{ni} is the normalising factor:

$$\gamma_{ni} = \exp[m^2\theta_i] + \sum_{k=0}^{m-1} \exp[k(2m-k)\,\theta_i]\,2\cosh[(m-k)(\beta_n-\delta_i)]. \tag{16}$$

The corresponding curves of the probabilistic functions are shown in Fig. 3.

It is evident in Fig. 3 that the probabilistic function for the most extreme positive response ($X_{ni}=m$) is single peaked and that the most extreme negative response ($X_{ni}=0$) is single troughed. The rest of the curves have two peaks. It was also understood that as a function of person locations, the mathematical expectation of the polytomous response variable is single peaked. However, the analytic properties of the undulating probabilistic functions for the ordered response categories, particularly the meaning of the intersection points of the adjacent response categories, need to be explored further with some more general forms.

Roberts and Laughlin (1996) developed the GUM (graded unfolding model) in a similar way except that the most positive response ($X_{ni}=m$) was also considered

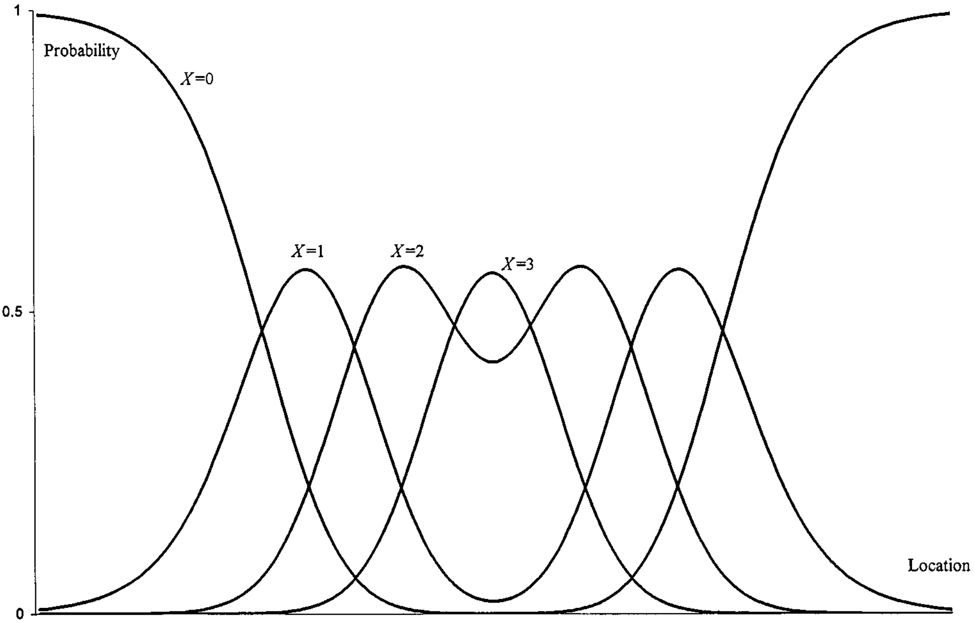

FIG. 3. The probabilistic functions of the GHCM.

to be two corresponding cumulative responses folded up. Therefore, the corresponding latent cumulative model has $2(m+1)$ categories. Derived from model (10), the probabilistic function for the GUM takes the form

$$\Pr\{X_{ni}=x\} = \frac{\exp[x(\beta_n-\delta_i)-\sum_{k=0}^{x}\tau_k]+\exp[(2m+1-x)(\beta_n-\delta_i)-\sum_{k=0}^{x}\tau_k]}{\left(\begin{array}{l}\sum_{l=0}^{m}\{\exp[l(\beta_n-\delta_i)-\sum_{k=0}^{l}\tau_k]\\ \qquad +\exp[(2m+1-l)(\beta_n-\delta_i)-\sum_{k=0}^{l}\tau_k]\}\end{array}\right)},$$

$$x=0, ..., m. \tag{17}$$

1.4. Approach and Structure of This Paper

This paper presents a formulation for a class of polytomous unfolding models with an attempt to uncover the hitherto implicit structure of unfolding models for ordered polytomous responses. The approach used in this paper does not follow Andrich (1996), Rost and Luo (1997), and Roberts and Laughlin (1996). Instead, by means of the rating formulation in a broader setting, a polytomous response which follows an unfolding process is also considered to be an observation of a random response vector, of which the components are dichotomous and all possible observations are in the collection of Guttman response patterns. Together with some particular attention to the special features of dichotomous unfolding models which distinguish them from the dichotomous cumulative models, a class of polytomous unfolding models is constructed. This class of polytomous unfolding models is sufficiently general that the models proposed by Andrich (1996), Rost and Luo (1997), and Roberts and Laughlin (1996) are special cases. Furthermore, some new models can be readily specified according to the general form of this class of the unfolding models for polytomous responses.

To obtain an insight into constructing models for ordered polytomous responses, the next section abstracts the approaches used in developing the Rasch models for polytomous responses into a broader setting. Under the context of a general response process, the mathematical equivalence between the rating formulation and Master's approach is formalised. Therefore, these approaches are also appropriate in the specified context of the unfolding response process. It is then shown that to construct a polytomous unfolding model in terms of dichotomous unfolding variables, it is operational to set the series of unfolding dichotomous variables with the same location while the values of their latitude of acceptance parameters vary. With this setting, a one-to-one correspondence is mapped between the manifest polytomous responses and the collection of Guttman patterns so that the probabilistic functions of the manifest polytomous responses are expressed in terms of the dichotomous unfolding response functions. It is then clarified that the models formulated are within the frame of unfolding models. Furthermore, Theorem 2 of Section 3 shows that the model has a desirable property when the latent dichotomous variables are in the order of the values for their latitude of acceptance parameters. It is also demonstrated that the unfolding models proposed in the literature can be considered special cases of the models formulated in this paper.

2. THE RATING FORMULATION IN GENERAL SITUATIONS

In Andrich's rating formulation, the key procedure to present the polytomous cumulative model is to set up a mapping between the ordered polytomous responses and a random vector of which the components are dichotomous and the co-domain Ω' is the collection of the Guttman patterns. This section shows that this procedure can be extended into a more general context. Then, in particular, the principle of the rating formulation can be applied to construct unfolding models.

2.1. General Process

In a general context, a polytomous response variable X with possible value $\{0, 1, ..., m\}$ can be mapped into an observation of a random dichotomous vector $Z = (Z_1, Z_2, ..., Z_m)$ as

$$(Z = (\underbrace{1, 1, ..., 1}_{k}, \underbrace{0, 0, ..., 0}_{m-k})) = (X = k), \qquad k = 0, 1, ..., m. \tag{18}$$

The collection of possible mapping observations of X to Z is $\Omega' = \{ V_k = (\underbrace{1, 1, ..., 1}_{k}, \underbrace{0, 0, ..., 0}_{m-k}), k = 0, ..., m\}$. It is evident the components of Z are not independent on Ω'. In particular,

$$\Pr\{Z_k = 1 \mid Z_{k+1} = 1\} = \Pr\{Z_{k+1} = 0 \mid Z_k = 0\} = 1, \qquad \text{for any} \quad k = 1, ..., m-1. \tag{19}$$

Let $\Delta = \{0, 1\}$. Then all the possible outcomes of $Z = (Z_1, Z_2, ..., Z_m)$ are $\Omega = \underbrace{\Delta \times \Delta \times \cdots \times \Delta}_{m}$. It is evident that $\Omega' \subset \Omega$. Let the marginal probabilities of Z_k on Ω be

$$\begin{aligned} p_k &\equiv \Pr\{Z_k = 1\}, \\ q_k &\equiv 1 - p_k = \Pr\{Z_k = 0\}, \qquad k = 1, ..., m. \end{aligned} \tag{20}$$

Then according to the definition of the conditional probability on $\Omega' \subset \Omega$,

$$\begin{aligned} \Pr\{Z = (\underbrace{1, 1, ..., 1}_{k}, \underbrace{0, 0, ..., 0}_{m-k}) \mid \Omega'\} &= \frac{\Pr\{Z = (\underbrace{1, 1, ..., 1}_{k}, \underbrace{0, 0, ..., 0}_{m-k}) \mid \Omega\}}{\Pr\{\Omega' \mid \Omega\}} \\ &= \frac{(\prod_{l=1}^{k} p_l)(\prod_{l=k+1}^{m} q_l)}{\sum_{j=0}^{m} (\prod_{l=1}^{j} p_l)(\prod_{l=j+1}^{m} q_l)} \\ &= \frac{(\prod_{l=1}^{k} p_l)(\prod_{l=k+1}^{m} q_l)}{\gamma}, \end{aligned} \tag{21}$$

where

$$\gamma = \sum_{j=0}^{m} \left(\prod_{l=1}^{j} p_l \right) \left(\prod_{l=j+1}^{m} q_l \right). \tag{22}$$

(When $j = 0$, $\prod_{l=1}^{j} p_l$ is defined as 1; when $j = m$, $\prod_{l=j+1}^{m} q_l$ is also defined as 1.)

According to (18), the probabilistic function of X takes the form

$$\begin{aligned} \Pr\{X = k\} &= \Pr\{Z = (\underbrace{1, 1, ..., 1}_{k}, \underbrace{0, 0, ..., 0}_{m-k}) \mid \Omega'\} \\ &= \frac{(\prod_{l=1}^{k} p_l)(\prod_{l=k+1}^{m} q_l)}{\gamma}, \qquad k = 0, ..., m. \end{aligned} \tag{23}$$

Conversely, $\{p_k, k = 1, ..., m\}$ can be expressed in terms of $\{\Pr\{X = k\}, k = 0, 1, ..., m\}$, as shown in the following.

LEMMA 1. *An equivalent expression of* (23) *is*

$$\begin{aligned} p_k &= \frac{\Pr\{X = k\}}{\Pr\{X = k-1\} + \Pr\{X = k\}}, \qquad k = 1, ..., m; \\ &\sum_{k=0}^{m} \Pr\{X = k\} = 1. \end{aligned} \tag{24}$$

Proof. From (23), for any k, $0 < k \leqslant m$, direct substitution leads to

$$\frac{\Pr\{X=k\}}{\Pr\{X=k-1\}+\Pr\{X=k\}}=\frac{(\prod_{l=1}^{k} p_l)(\prod_{l=k+1}^{m} q_l)}{(\prod_{l=1}^{k-1} p_l)(\prod_{l=k}^{m} q_l)+(\prod_{l=1}^{k} p_l)(\prod_{l=k+1}^{m} q_l)}$$

$$=\frac{p_k}{q_k+p_k}=p_k. \tag{25}$$

Together with (22) by which the denominator γ is defined, (25) leads to (24). Conversely, from (24), for any k, $0 < k \leqslant m$,

$$\prod_{l=1}^{k} p_l \prod_{l=k+1}^{m} q_l=\left(\prod_{l=1}^{k}\frac{\Pr\{X=l\}}{\Pr\{X=l-1\}+\Pr\{X=l\}}\right)$$
$$\times\left(\prod_{l=k+1}^{m}\frac{\Pr\{X=l-1\}}{\Pr\{X=l-1\}+\Pr\{X=l\}}\right)$$
$$=\frac{(\prod_{l=1}^{k}\Pr\{X=l\})(\prod_{l=k+1}^{m}\Pr\{X=l-1\})}{\coprod_{l=1}^{m}[\Pr\{X=l-1\}+\Pr\{X=l\}]}$$
$$=\Pr\{X=k\}\cdot\frac{\prod_{l=1}^{m-1}\Pr\{X=l\}}{\coprod_{l=1}^{m}[\Pr\{X=l-1\}+\Pr\{X=l\}]}. \tag{26}$$

Define γ according to

$$\gamma\equiv\frac{\prod_{l=1}^{m-1}\Pr\{X=l\}}{\coprod_{l=1}^{m}[\Pr\{X=l-1\}+\Pr\{X=l\}]}=\frac{\prod_{l=1}^{m} p_l}{\Pr\{X=m\}}. \tag{27}$$

It is evident that γ is not related with k. Therefore, (24) leads to (23). ∎

It is noted that though (23) and (24) are equivalent, the explicit focus of each is different. Equation (23) focuses on the overall structure of the latent dichotomous responses while (24) focuses on the pairs of adjacent response categories. In the context of Rasch models, Andrich (1978) derived model (8) in the form of (23). Masters (1982) began with (24) and obtained model (13). In Andrich's approach, the requirement on the ordering of the dichotomous variables is essential. Andrich (1985) provided further elaboration on the importance of this requirement. The author of this paper considers that when a polytomous response is mapped into a set of dichotomous responses, the relationship between these dichotomous responses cannot be arbitrary. Though the ordering requirement is not the mathematical property of the probabilistic function of model (11), it is the requirement on the structure of the model.

2.2. *Unfolding Process*

In particular, according to Lemma 1, an unfolding model for polytomous responses, which is the main interest of this paper, should lead to a dichotomous model when the person is asked to give his or her preference on the pairs of the adjacent categories. In this case, the closer the person's location to the item's, the more likely the higher or more positive response category should be chosen. When the responses are coded as 1, the higher response category is chosen, and 0, the lower response category is chosen, then this dichotomous response process should be single peaked around the location of the polytomous item being considered. It implies that the preferences of the adjacent categories are anchored at the location δ of the item invoking a polytomous response. That is, the dichotomised items arising from adjacent category preference have the same locations as the manifest item. In addition, these dichotomous responses should follow some pattern. For instance, when the response categories are 0, *Strongly Disagree*, 1, *Strongly Disagree*, 2, *Agree*, and 3, *Strongly Agree*, if a person chose *Strongly Agree* out of the adjacent categories *Strongly Agree* and *Agree*, then the person is expected with certainty to choose *Agree* out of the adjacent categories *Agree* and *Disagree*. In general, the joint sample space of these dichotomous items is Ω', the collection of the Guttman patterns, as defined in the previous section.

It is noted in the general form (4) of unfolding models for dichotomous responses that for an item which follows the dichotomous unfolding process, its location parameter δ_i together with its latitude of acceptance parameter ρ_i govern the response process. The parameter ρ_i is a threshold: when the person–item distance is smaller than $|\beta_n - \delta_i| < \rho_i$, a positive response is more likely; otherwise, when $|\beta_n - \delta_i| > \rho_i$, a negative response is more likely. This biparameterisation of the dichotomous unfolding models makes it possible that a set of dichotomous unfolding variables have the same location but different latitudes of acceptance.

Figure 4 shows that for each dichotomous unfolding variable Z_k, its location δ_k and latitude of acceptance ρ_k define an interval $[\delta_k - \rho_k, \delta_k + \rho_k]$ in which a positive response is more likely. Outside this interval, a negative response is more

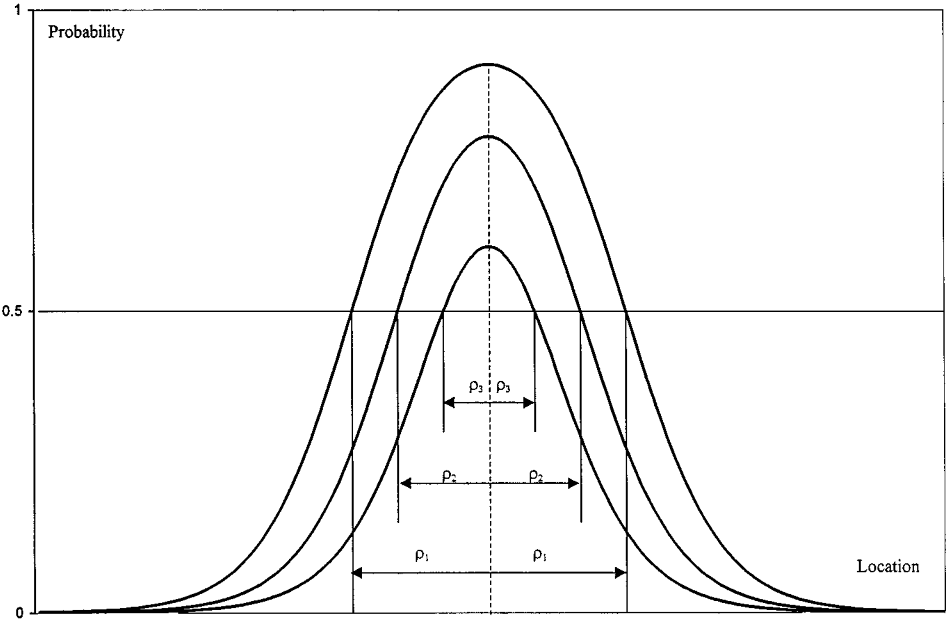

FIG. 4. The probabilistic functions of the general unfolding model for dichotomous responses.

likely. Figure 4 also shows that for the response variables which have the same value for their locations, the greater the value of ρ_k, the greater the range of persons on the continuum who find the higher order response acceptable.

It will be seen in the next section that the requirement that the dichotomous variables have the same location provides a base on which a general form of unfolding models for polytomous responses is constructed.

3. CONSTRUCTING THE CLASS OF POLYTOMOUS UNFOLDING MODELS

3.1. *Presentation of the Probabilistic Functions*

Again, consider that a person n with location β_n is invited to give a response to a statement (item) i with the following common response alternatives:

Strongly disagree	Disagree	Agree	Strongly agree
0	1	2	3

The rating formulation approach generally maps a manifest polytomous response $X_{ni} \in \{0, 1, 2, 3\}$ to three dichotomous variables $(Z_{ni1}, Z_{ni2}, Z_{ni3})$. In the context of the unfolding process as discussed in the previous section, it is central to suppose that the dichotomous variables for the same polytomous variable follow the general form of the unfolding models with a different latitude of acceptance parameters $\{\rho_{ik}\}$ and the same location parameter δ_i. Under this setting, for $k = 1, 2, 3$,

$$p_{nik}(\rho_{ik}, \beta_n, \delta_i) \equiv \Pr\{Z_{nik} = 1 \mid \rho_{ik}, \beta_n, \delta_i\} = \frac{\Psi_k(\rho_{ik})}{\Psi_k(\rho_{ik}) + \Psi_k(\beta_n - \delta_i)},$$
$$q_{nik}(\rho_{ik}, \beta_n, \delta_i) \equiv \Pr\{Z_{nik} = 0 \mid \rho_{ik}, \beta_n, \delta_i\} = \frac{\Psi_k(\beta_n - \delta_i)}{\Psi_k(\rho_{ik}) + \Psi_k(\beta_n - \delta_i)}. \quad (28)$$

Therefore, according to (23), the probability for the values of X can be expressed as the conditional probability of the corresponding pattern in Ω':

$$\Pr\{X_{ni} = 0\} = \frac{1}{\gamma_{ni}} q_{ni1} q_{ni2} q_{ni3} = \frac{\Psi_1(\beta_n - \delta_i)\, \Psi_2(\beta_n - \delta_i)\, \Psi_3(\beta_n - \delta_i)}{\gamma_{ni} \prod_{k=1}^{3} [\Psi_k(\rho_{ik}) + \Psi_k(\beta_n - \delta_i)]},$$
$$\Pr\{X_{ni} = 1\} = \frac{1}{\gamma_{ni}} (p_{ni1} q_{ni2} q_{ni3}) = \frac{\Psi_1(\rho_{i1})\, \Psi_2(\beta_n - \delta_i)\, \Psi_3(\beta_n - \delta_i)}{\gamma_{ni} \prod_{k=1}^{3} [\Psi_k(\rho_{ik}) + \Psi_k(\beta_n - \delta_i)]},$$
$$\Pr\{X_{ni} = 2\} = \frac{1}{\gamma_{ni}} (p_{ni1} p_{ni2} q_{ni3}) = \frac{\Psi_1(\rho_{i1})\, \Psi_2(\rho_{i2})\, \Psi_3(\beta_n - \delta_i)}{\gamma_{ni} \prod_{k=1}^{3} [\Psi_k(\rho_{ik}) + \Psi_k(\beta_n - \delta_i)]},$$
$$\Pr\{X_{ni} = 3\} = \frac{1}{\gamma_{ni}} (p_{ni1} p_{ni2} p_{ni3}) = \frac{\Psi_1(\rho_{i1})\, \Psi_2(\rho_{i2})\, \Psi_3(\rho_{i3})}{\gamma_{ni} \prod_{k=1}^{3} [\Psi_k(\rho_{ik}) + \Psi_k(\beta_n - \delta_i)]}, \quad (29)$$

were

$$\gamma_{ni} = \frac{1}{\prod_{k=1}^{3} [\Psi_k(\rho_{ik}) + \Psi_k(\beta_n - \delta_i)]} \{\Psi_1(\beta_n - \delta_i)\, \Psi_2(\beta_n - \delta_i)\, \Psi_3(\beta_n - \delta_i)$$
$$+ \Psi_1(\rho_{i1})\, \Psi_2(\beta_n - \delta_i)\, \Psi_3(\beta_n - \delta_i)$$
$$+ \Psi_1(\rho_{i1})\, \Psi_2(\rho_{i2})\, \Psi_3(\beta_n - \delta_i)$$
$$+ \Psi_1(\rho_{i1})\, \Psi_2(\rho_{i2})\, \Psi_3(\rho_3)\}. \tag{30}$$

In general, Eq. (29) holds when the maximum score of X_{ni} is a positive integer m and the alternatives of a response are $\{0, 1, 2, ..., m\}$. Then in terms of the probabilistic functions of the dichotomous variables of Eq. (4), the expression for the probabilistic functions of the manifest variable is

$$\Pr\{X_{ni} = k\} = \frac{\left(\prod_{l=1}^{k} p_{nil}\right)\left(\prod_{l=k+1}^{m} q_{nil}\right)}{\gamma_{ni}}$$
$$= \frac{\left(\prod_{l=1}^{k} \frac{\Psi_l(\rho_{il})}{\Psi_l(\rho_{il}) + \Psi(\beta_n - \delta_i)}\right)\left(\prod_{l=k+1}^{m} \frac{\Psi_l(\beta_n - \delta_i}{\Psi_l(\rho_{il}) + \Psi_l(\beta_n - \delta_i)}\right)}{\gamma_{ni}}$$
$$= \frac{\left(\prod_{l=1}^{k} \Psi_l(\rho_{il})\right)\left(\prod_{l=k+1}^{m} \Psi_l(\beta_n - \delta_i)\right)}{\gamma_{ni} \prod_{l=1}^{m} [\Psi_l(\rho_{il}) + \Psi_l(\beta_n - \delta_i)]}, \qquad k = 0, ..., m, \tag{31}$$

where γ_{ni} is a normalising factor

$$\gamma_{ni} = \sum_{k=0}^{m} \left(\prod_{l=1}^{k} \frac{\Psi_l(\rho_{il})}{\Psi_l(\rho_{il}) + \Psi_l(\beta_n - \delta_i)}\right)\left(\prod_{l=k+1}^{m} \frac{\Psi_l(\beta_n - \delta_i)}{\Psi_l(\rho_{il}) + \Psi_l(\beta_n - \delta_i)}\right). \tag{32}$$

Letting

$$\lambda_{ni} \equiv \gamma_{ni} \prod_{l=1}^{m} [\Psi_l(\rho_{il}) + \Psi_l(\beta_n - \delta_i)] = \sum_{k=0}^{m} \left(\prod_{l=1}^{k} \Psi_l(\rho_{il})\right)\left(\prod_{l=k+1}^{m} \Psi_l(\beta_n - \delta_i)\right), \tag{33}$$

we obtain a formulation of polytomous unfolding models as the following

$$\Pr\{X_{ni} = k \mid \beta_n, \delta_i, (\rho_{il})\} = \frac{(\prod_{l=1}^{k} \Psi_l(\rho_{il}))(\prod_{l=k+1}^{m} \Psi_l(\beta_n - \delta_i))}{\lambda_{ni}}, \qquad k = 0, ..., m. \tag{34}$$

3.2. Confirmation of the Unfolding Process

THEOREM 1. *The expectation of the class of models* (34) *is a single peaked function of the person location parameter. That is, for any m, when* $|\beta'_n-\delta_i|\geqslant|\beta_n-\delta_i|$,

$$E(X_{ni}\mid\beta_n)=\frac{\sum_{k=0}^{m}k(\prod_{l=1}^{k}\Psi_l(\rho_{il}))(\prod_{l=k+1}^{m}\Psi_l(\beta_n-\delta_i))}{\sum_{k=0}^{m}(\prod_{l=1}^{k}\Psi_l(\rho_{il}))(\prod_{l=k+1}^{m}\Psi_l(\beta_n-\delta_i))}$$

$$\geqslant\frac{\sum_{k=0}^{m}k(\prod_{l=1}^{k}\Psi_l(\rho_{il}))(\prod_{l=k+1}^{m}\Psi_l(\beta'_n-\delta_i))}{\sum_{k=0}^{m}(\prod_{l=1}^{k}\Psi_l(\rho_{il}))(\prod_{l=k+1}^{m}\Psi_l(\beta'_n-\delta_i))}=E(X_{ni}\mid\beta'_n). \tag{35}$$

Proof. The theorem is proved by induction. First for any m and any $1\leqslant j\leqslant m$, define

$$E_j(X_{ni})=\frac{\sum_{k=0}^{j}k(\prod_{l=1}^{k}\Psi_l(\rho_{il}))(\prod_{l=k+1}^{j}\Psi_l(\beta_n-\delta_i))}{\lambda_{nij}}, \tag{36}$$

where

$$\lambda_{nij}=\sum_{k=0}^{j}\left(\prod_{l=1}^{k}\Psi_l(\rho_{il})\right)\left(\prod_{l=k+1}^{j}\Psi_l(\beta_n-\delta_i)\right). \tag{37}$$

Because all operational functions are nonnegative,

$$E_j(X_{ni})=\frac{\sum_{k=0}^{j}k(\prod_{l=1}^{k}\Psi_l(\rho_{il}))(\prod_{l=k+1}^{j}\Psi_l(\beta_n-\delta_i))}{\lambda_{nij}}$$

$$\leqslant\frac{j\sum_{k=0}^{j}(\prod_{l=1}^{k}\Psi_l(\rho_{il}))(\prod_{l=k+1}^{j}\Psi_l(\beta_n-\delta_i))}{\lambda_{nij}}=j. \tag{38}$$

The first step of induction requires one to examine the validity when $j=1$. In fact,

$$E_1(X_{ni})=\frac{\Psi_1(\rho_{i1})}{\Psi_1(\rho_{i1})+\Psi_1(\beta_n-\delta_i)}. \tag{39}$$

That is, $E_1(X_{ni})$ is the expectation of the dichotomous unfolding variable Z_{ni1}, which is a single-peaked function of person location β_n. Then the assumption of the induction is that, for $j<m$, $E_j(X_{ni})$ is a single-peaked function of person location β_n. That is, when $|\beta'_n-\delta_i|\geqslant|\beta_n-\delta_i|$,

$$E_j(X_{ni})\geqslant E'_j(X_{ni}) \tag{40}$$

where $E'_j(X_{ni})$ is defined as in (36) when β_n is replaced with β'_n. The following equation expresses $E_{j+1}(X_{ni})$ in terms of $E_j(X_{ni})$:

$$E_{j+1}(X_{ni}) = \frac{\sum_{k=0}^{j+1} k(\prod_{l=1}^{k} \Psi_l(\rho_{il}))(\prod_{l=k+1}^{j+1} \Psi_l(\beta_n - \delta_i))}{\sum_{k=0}^{j+1} (\prod_{l=1}^{k} \Psi_l(\rho_{il}))(\prod_{l=k+1}^{j+1} \Psi_l(\beta_n - \delta_i))}$$
$$= \frac{\begin{pmatrix}(j+1)(\prod_{l=1}^{j+1} \Psi_l(\rho_{il})) + \Psi_{j+1}(\beta_n - \delta_i) \sum_{k=0}^{j} k \\ (\prod_{l=1}^{k} \Psi_l(\rho_{il}))(\prod_{l=k+1}^{j} \Psi_l(\beta_n - \delta_i))\end{pmatrix}}{\begin{pmatrix}\prod_{l=1}^{j+1} \Psi_l(\rho_{il}) + \Psi_{j+1}(\beta_n - \delta_i) \sum_{k=0}^{j} \\ (\prod_{l=1}^{k} \Psi_l(\rho_{il}))(\prod_{l=k+1}^{j} \Psi_l(\beta_n - \delta_i))\end{pmatrix}}$$
$$= \frac{(j+1) \prod_{l=1}^{j+1} \Psi_l(\rho_{il}) + \Psi_{j+1}(\beta_n - \delta_i)\, E_j(X_{ni})\, \lambda_{nij}}{\prod_{l=1}^{j+1} \Psi_l(\rho_{il}) + \Psi_{j+1}(\beta_n - \delta_i)\, \lambda_{nij}}. \tag{41}$$

According to (38),

$$j+1 \geqslant E_j(X_{ni}) > 0. \tag{42}$$

Because $\{\Psi_l\}$ are nonnegative functions and

$$\Psi_{j+1}(\beta_n - \delta_i)\, \lambda_{nij} \geqslant 0, \tag{43}$$

we have

$$1 \geqslant \alpha_j \equiv \frac{\prod_{l=1}^{j} \Psi_l(\rho_{il})}{\prod_{l=1}^{j} \Psi_l(\rho_{il}) + \Psi_{j+1}(\beta_n - \delta_i)\, \lambda_{nij}} \geqslant 0. \tag{44}$$

Therefore, $E_{j+1}(X_{ni})$ can be expressed as a weighted sum of $(j+1)$ and $E_j(X_{ni})$:

$$E_{j+1}(X_{ni}) = (j+1)\, \alpha + E_j(X_{ni})(1-\alpha). \tag{45}$$

Furthermore, because when $|\beta_n - \delta_i| < |\beta'_n - \delta_i|$, $\Psi_l(\beta_n - \delta_i) < \Psi_l(\beta'_n - \delta_i)$, $l = 1, 2, ..., m$, then

$$\lambda_{nij} = \sum_{k=0}^{j} \left(\prod_{l=1}^{k} \Psi_l(\rho_{il})\right)\left(\prod_{l=k+1}^{j} \Psi_l(\beta_n - \delta_i)\right)$$
$$\leqslant \sum_{k=0}^{j} \left(\prod_{l=1}^{k} \Psi_l(\rho_{il})\right)\left(\prod_{l=k+1}^{j} \Psi_l(\beta'_n - \delta_i)\right) \equiv \lambda'_{nij}, \tag{46}$$

and

$$1 \geqslant \alpha_j = \frac{\prod_{l=1}^{k} \Psi_l(\rho_{il})}{\prod_{l=1}^{k} \Psi_l(\rho_{il}) + \Psi_m(\beta_n - \delta_i)\, \lambda_{ni(m-1)}}$$
$$\geqslant \frac{\prod_{l=1}^{k} \Psi_l(\rho_{il})}{\prod_{l=1}^{k} \Psi_l(\rho_{il}) + \Psi_m(\beta'_n - \delta_i)\, \lambda'_{ni(m-1)}} \equiv \alpha'_j \geqslant 0. \tag{47}$$

According to the assumption of the induction (40), we have

$$E_j(X_{ni}) \geqslant E'_j(X_{ni}). \tag{48}$$

Then (42), (47), and (48) lead to

$$\begin{aligned} E_{j+1}(X_{ni}) &= (j+1)\,\alpha_j + E_j(X_{ni})(1-\alpha_j) \\ &= \alpha_j[(j+1) - E_j(X_{ni})] + E_j(X_{ni}) \\ &\geqslant \alpha'_j[(j+1) - E_j(X_{ni})] + E_j(X_{ni}) \\ &= (j+1)\,\alpha'_j + E_j(X_{ni})(1-\alpha'_j) \\ &\geqslant (j+1)\,\alpha'_j + E'_j(X_{ni})(1-\alpha'_j) \\ &= E'_{j+1}(X_{ni}). \end{aligned} \tag{49}$$

Then (40) holds for any $j = 1, 2, ..., m$. In particular, when $j = m-1$, (35) holds. ∎

3.3. Analytic Properties of the Probabilistic Functions

For each latent dichotomous variable Z_k, if and only if $|\beta_n - \delta_i| = \rho_k$,

$$p_{nik} = \Pr\{Z_k = 1\} = \Pr\{Z_k = 0\} = q_{nik} = \tfrac{1}{2}. \tag{50}$$

Then according to Lemma 1, when (50) holds,

$$\Pr\{X = k-1\} = \Pr\{X = k\}. \tag{51}$$

Conversely, according to Lemma 1, Eq. (51) also implies (50) or equivalently, $|\beta_n - \delta_i| = \rho_k$.

The equivalence of Eqs. (50) and (51) is significant in interpreting the graphic behaviour of the probabilistic functions. It reveals that the crossing points of the probabilistic functions of the adjacent categories are the values of the latitudes of acceptance parameters of the corresponding latent dichotomous variables. To show this point, Fig. 5 plots the probabilistic functions of the model (70) together with the probabilistic functions of the corresponding dichotomous variables ($m = 3$).

Furthermore, Fig. 5 shows that if the thresholds are ordered in their values, then for any $k > 0$, in the intervals $[\delta_i + \rho_{k+1}, \delta_i + \rho_k]$ and $[\delta_i - \rho_k, \delta_i - \rho_{k+1}]$, the probability $\Pr\{X = k\}$ has the greatest value among call categories. That is, within these intervals, $\{X = k\}$ is the most likely. The following theorem confirms that the model (34) has the desirable property when the latent dichotomous variables are in the order of the values of their latitude of acceptance parameters.

THEOREM 2. *Suppose that $\rho_1 > \rho_2 > \cdots > \rho_k > \cdots > \rho_m$. For $m > k_0 > 0$ and any β_n, if $\rho_{k_0} > |\beta_n - \delta_i| > \rho_{k_0+1}$, then for any k,*

$$P\{X = k_0 \mid \beta_n\} \geqslant P\{X = k \mid \beta_n\}. \tag{52}$$

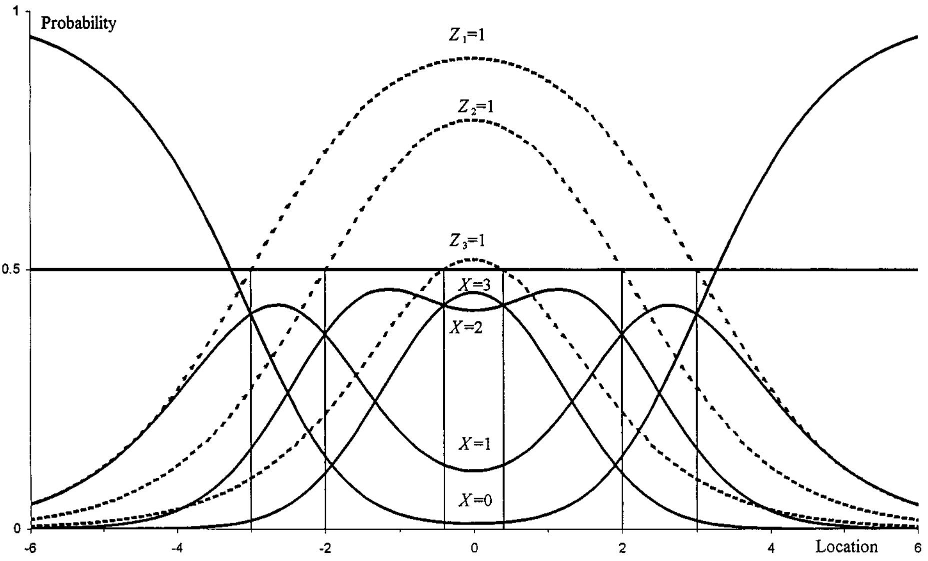

FIG. 5. The probabilistic functions of the unfolding model GUM for polytomous responses.

Proof. According to (34),

$$\Pr\{X=k_0 \mid \beta_n\} = \frac{(\prod_{l=1}^{k_0} p_l)(\prod_{l=k_0+1}^{m} q_l)}{\lambda}, \tag{53}$$

$$\Pr\{X=k \mid \beta_n\} = \frac{(\prod_{l=1}^{k} p_l)(\prod_{l=k+1}^{m} q_l)}{\lambda}. \tag{54}$$

Consider their ratio

$$R = \frac{\Pr\{X=k_0 \mid \beta_n\}}{\Pr\{X=k \mid \beta_n\}} = \frac{(\prod_{l=1}^{k_0} p_l)(\prod_{l=k_0+1}^{m} q_l)}{(\prod_{l=1}^{k} p_l)(\prod_{l=k+1}^{m} q_l)}. \tag{55}$$

The following is to prove that $R > 1$, which leads to (52). When $k > k_0$,

$$R = \frac{\prod_{l=k_0+1}^{k} q_l}{\prod_{l=k_0+1}^{k}} = \prod_{l=k_0+1}^{k} \frac{q_l}{p_l}. \tag{56}$$

Because $|\beta_n - \delta_i| > \rho_{k_0+1}$ and for any $l > k_0$, $|\beta_n - \delta_i| > \rho_{k_0+1} \geqslant \rho_l$,

$$q_l = \Pr\{Z_l = 0\} > \tfrac{1}{2} > \Pr\{Z_l = 1\} = p_l.$$

Therefore, $R > 1$. When $k < k_0$,

$$R = \frac{\prod_{l=k+1}^{k_0} p_l}{\prod_{l=k+1}^{k_0} q_l} = \prod_{l=k+1}^{k_0} \frac{p_l}{q_l}. \tag{57}$$

Because $|\beta_n - \delta_i| < \rho_{k_0}$ and for any $l < k_0$, $|\beta_n - \delta_i| < \rho_{k_0} \leqslant \rho_l$,

$$q_l = \Pr\{Z_l = 0\} < \tfrac{1}{2} < \Pr\{Z_l = 1\} = p_l,$$

which leads to $R > 1$.

3.4. A Special Case: The GUM

The GUM (Roberts & Laughlin, 1996) is a model for polytomous unfolding responses, developed by folding the Rasch model for polytomous responses. If the unfolding responses have the maximum score of m, then the probabilistic function of the GUM can be derived as (17), which was expressed in the beginning of this paper. This section will demonstrate that if the corresponding Rasch model for polytomous responses for Eq. (17) has ordered thresholds

$$\tau_1 < \tau_2 < \cdots < \tau_{2m} < \tau_{2m+1}, \tag{58}$$

then (17) is a special case of the general unfolding model of (34).

First, it is noted that the symmetric requirement on (17) in Roberts and Laughlin (1996) is

$$\tau_k = \tau_{2m+2-k}, \qquad k = 1, ..., m. \tag{59}$$

Under the condition of (58), (59) leads to $\tau_k \leqslant 0$ or, equivalently,

$$\exp(-\tau_k) \geqslant 1. \tag{60}$$

For $k = 1, 2, ..., m$, according to (17), After some simplification, we have

$$\begin{aligned}
&\frac{\Pr\{X_{ni} = k\}}{\Pr\{X_{ni} = k-1\} + \Pr\{X_{ni} = k\}} \\
&\quad = \frac{\{\exp[k\mu] + \exp[(2m+1-k)\mu]\}}{\begin{pmatrix}\exp[\tau_k]\{\exp[(k-1)\mu] + \exp[\{2m+1-(k-1)\}\mu]\} \\ + \{\exp[k\mu] + \exp[(2m+1-k)\mu]\}\end{pmatrix}} \\
&\quad = \frac{\cosh\left[\left(k - \frac{2m+1}{2}\right)\mu\right]}{\exp[\tau_k]\left\{\cosh\left[\left\{(k-1) - \frac{2m+1}{2}\right\}\mu\right]\right\} + \cosh\left[\left(k - \frac{2m+1}{2}\right)\mu\right]} \\
&\quad = \frac{\exp[-\tau_k]}{\dfrac{\cosh\left[\left(\frac{2m+1}{2} + 1 - k\right)\mu\right]}{\cosh\left[\left(\frac{2m+1}{2} - k\right)\mu\right]} + \exp[-\tau_k]},
\end{aligned} \tag{61}$$

where

$$\mu = (\beta_n - \delta_i). \tag{62}$$

Let

$$\Psi_k(t) = \frac{\cosh\left[\left(\frac{2m+1}{2} + 1 - k\right) t\right]}{\cosh\left[\left(\frac{2m+1}{2} - k\right) t\right]}. \tag{63}$$

Then it is straightforward to show that $\Psi_k(t)$ satisfies the properties of (P1), (P2), and (P3). Select $\rho_{ik}(\geqslant 0)$ so that

$$\frac{\cosh\left[\left(\frac{2m+1}{2} + 1 - k\right) \rho_{ik}\right]}{\cosh\left[\left(\frac{2m+1}{2} - k\right) \rho_{ik}\right]} = \exp[-\tau_k]. \tag{64}$$

Then $\rho_{ik}(\geqslant 0)$ is well defined according to (60). Equation (61) is a probabilistic function of an unfolding model:

$$\frac{\Pr\{X_{ni}=k\}}{\Pr\{X_{ni}=k-1\} + \Pr\{X_{ni}=k\}} = \frac{\Psi_k(\rho_{ik})}{\Psi_k(\rho_{ik}) + \Psi_k(\beta_n - \delta_i)} = \Pr\{z_{nik}=1 \mid \beta_n, \delta_i, \rho_{ik}\}. \tag{65}$$

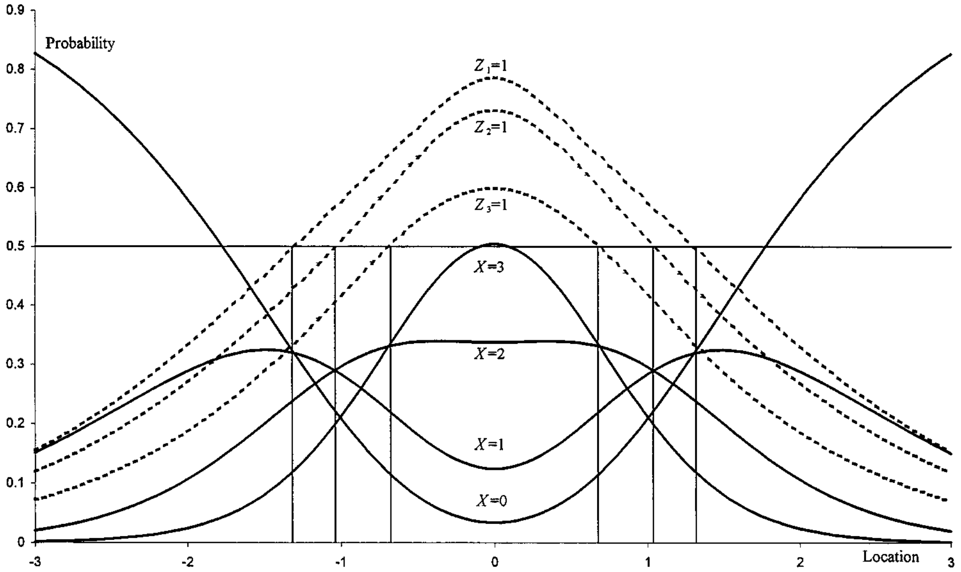

FIG. 6. The probabilistic functions of the unfolding model GUM for polytomous responses.

According to Lemma 1 in the previous section, the GUM can be reexpressed in the form of Eq. (34). Figure 6 shows the probabilistic functions of the GUM and the corresponding dichotomous unfolding variables.

A similar procedure can also show that the GHCM (Andrich, 1996 Rost & Luo, 1997) conforms to the general form (34) of the unfolding models for polytomous responses.

4. GENERATING NEW POLYTOMOUS UNFOLDING MODELS

From the general form (34) of the class of the unfolding models for polytomous responses, it is possible to specify various unfolding models based on different unfolding models for dichotomous responses. To simplify the expressions, however, the following examples take the same operational functions for all the dichotomous latent unfolding variables. The dichotomous unfolding models used in this section are the extended forms of the respective original models for dichotomous responses generalised in Luo (1998a).

4.1. The Simple Square Logistic Model (SSLM) for Polytomous Responses: SSLMP

Let (Andrich, 1988)

$$\Pr\{Z_{nik}=1\} = \frac{\exp(\rho_{ik}^2)}{\exp(\rho_{ik}^2)+\exp[(\beta_n-\delta_i)^2]}, \qquad k=1, ..., m. \tag{66}$$

Then according to (34),

$$\Pr\{X_{ni}=k\} = \frac{\exp\{\sum_{l=1}^{k} \rho_{il}^2\} \exp\{(m-k)(\beta_n-\delta_i)^2\}}{\lambda_{ni}},$$

$$k=0, 1, ..., m-1, \tag{67}$$

where

$$\lambda_{ni} = \sum_{k=0}^{m} \exp\left\{\sum_{l=1}^{k} \rho_{il}^2\right\} \exp\{(m-k)(\beta_n-\delta_i)^2\}. \tag{68}$$

Figure 7 shows the curves of the probabilistic functions of the SSLMP.

4.2. HCM for Polytomous Responses: HCMP

Let (Andrich & Luo, 1993; Luo 1998a)

$$\Pr\{Z_{nik}=1\} = \frac{\cosh(\rho_{ik})}{\cosh(\rho_{ik})+\cosh(\beta_n-\delta_i)}, \qquad k=1, ..., m. \tag{69}$$

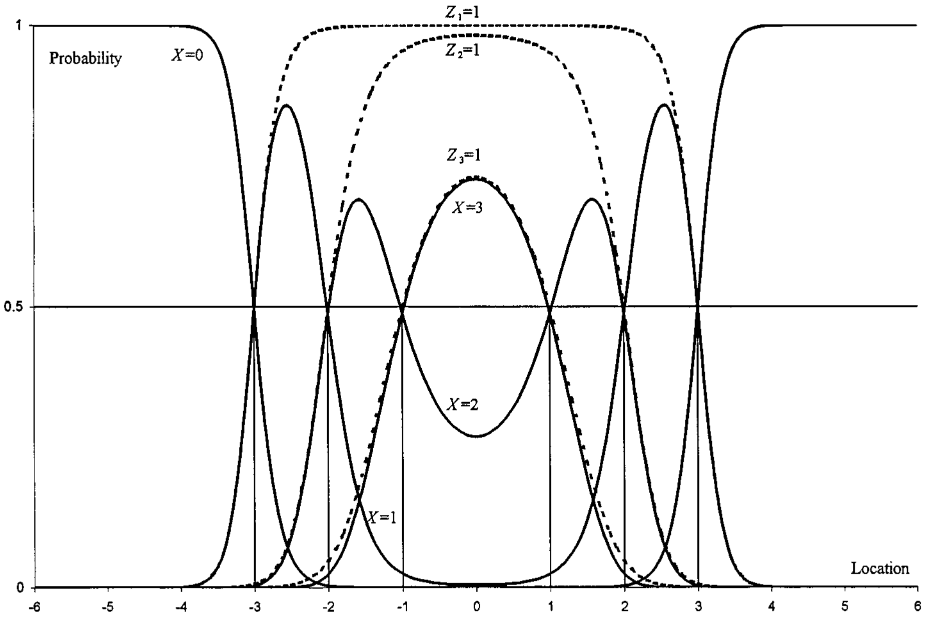

FIG. 7. The probabilistic functions of the unfolding model SSLMP for polytomous responses.

Then according to (34),

$$\Pr\{X_{ni}=k\} = \frac{[\cosh(\beta_n-\delta)]^{m-k}\prod_{l=1}^{k}\cosh(\rho_{il})}{\lambda_{ni}},$$

$$k=0, 1, ..., m-1, \tag{70}$$

where

$$\lambda_{ni} = \sum_{k=0}^{m}[\cosh(\beta_n-\delta)]^{m-k}\prod_{l=1}^{k}\cosh(\rho_{il}). \tag{71}$$

The curves of the probabilistic functions of the HCMP were shown in Fig. 5.

4.3. The PARELLA Model for Polytomous Responses: PARELLAP

Let (Hoijtink, 1990)

$$\Pr\{Z_{nik}=1\} = \frac{\rho_{ik}^2}{\rho_{ik}^2+(\rho_n-\delta_i)^2}, \qquad k=1, ..., m. \tag{72}$$

Then according to (34),

$$\Pr\{X_{ni}=k\} = \frac{(\beta_n-\delta)^{2(m-k)}\prod_l^k \rho_{ik}^2}{\lambda_{ni}}, \qquad k=0, 1, ..., m-1, \tag{73}$$

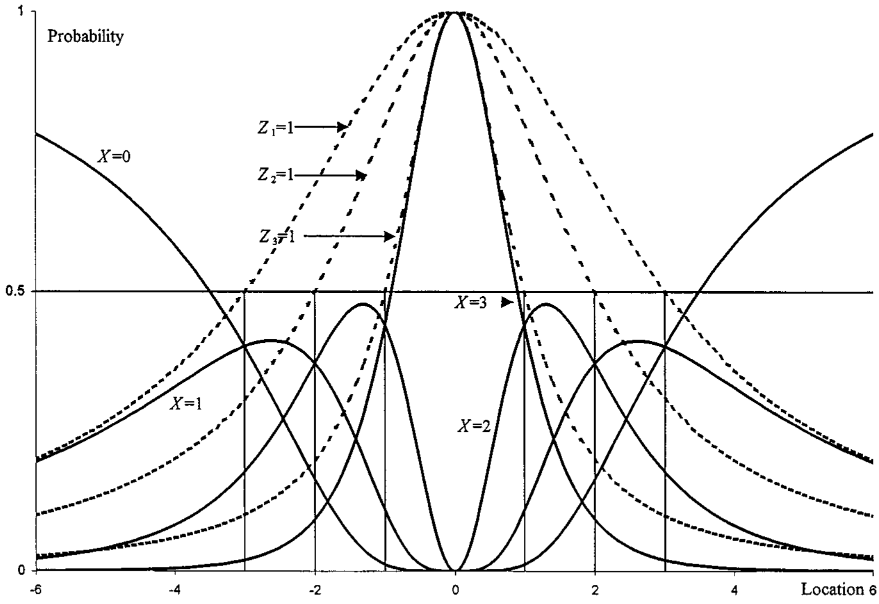

FIG. 8. The probabilistic functions of the unfolding model PARELLAP for polytomous responses.

where

$$\lambda_{ni} = \sum_{k=0}^{m} (\beta_n - \delta)^{2(m-k)} \prod_{l=1}^{k} \rho_{ik}^2. \tag{74}$$

(For notational convenience, $(\beta_n - \delta)^{2(m-k)} = 1$ when $k = m$ regardless of the value of $(\beta_n - \delta)$.) Figure 8 shows the curves of the probabilistic functions of the PARELLAP.

5. SUMMARY AND DISCUSSION

In the class of unfolding models proposed in this paper, the thresholds, which are the points of intersection of probabilistic functions for the adjacent response categories, are interpreted as the latitudes of acceptance parameters of the latent dichotomous unfolding variables which have the same location and jointly define the manifest polytomous response variable. Though the formulation of the general form of this class of unfolding models does not mathematically rely on the requirement that the latent dichotomous variables are in the order of their latitudes of acceptance, this requirement ensures that this class of models is well structured and conforms to the psychological mechanism by which the polytomous responses are made. The presentation of this class of unfolding models is focused on unidimensional situations. However, the result of this paper can be extended to the multidimensional situations, as in the dichotomous models (Luo, 1998b), provided the person–item distance in the general form of this class of unfolding models is considered as in a multidimensional space. The operationalization of this class of unfolding models, including parameter estimation and test of fit, is expected to facilitate the applications of a range of specific unfolding models in real psychological measurement (Luo, 1999).

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